Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2018

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2018/62748

Section Two: Calculator-assumed **65% (91 Marks)**

Question 10 **(4 marks)**

Consider the complex number $z = \sqrt{3} + i$.

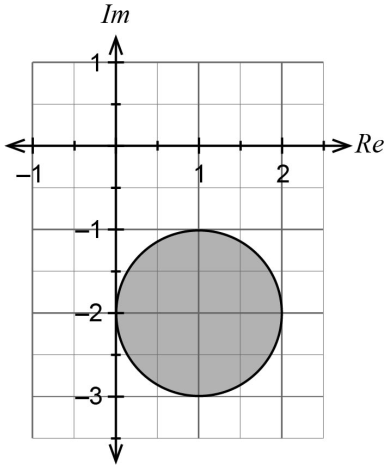
Show that, for all positive integers n, $(z)^n - (\bar{z})^n = 2^{n+1} \sin\left(\frac{n\pi}{6}\right) i$.

Solution
$z = \sqrt{3} + i = 2cis\left(\frac{\pi}{6}\right)$ $(z)^n - (\bar{z})^n = \left(2cis\left(\frac{\pi}{6}\right)\right)^n - \left(2cis\left(-\frac{\pi}{6}\right)\right)^n$ $= 2^n cis\left(\frac{n\pi}{6}\right) - 2^n cis\left(-\frac{n\pi}{6}\right)$ $= 2^n\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right) - 2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$ $= 2^n \cos\left(\frac{n\pi}{6}\right) + i\,2^n \sin\left(\frac{n\pi}{6}\right) - \left(2^n \cos\left(\frac{n\pi}{6}\right) - i\,2^n \sin\left(\frac{n\pi}{6}\right)\right)$ $= i\; 2.2^n \sin\left(\frac{n\pi}{6}\right) \quad \ldots (1)$ $= 2^{n+1} \sin\left(\frac{n\pi}{6}\right) i$
Specific behaviours
✓ expresses both z and $\bar{z}$ correctly in polar form ✓ uses de Moivre's Theorem correctly ✓ applies the parity of the cosine and sine functions correctly ✓ simplifies to the expression (1)

Question 11 **(8 marks)**

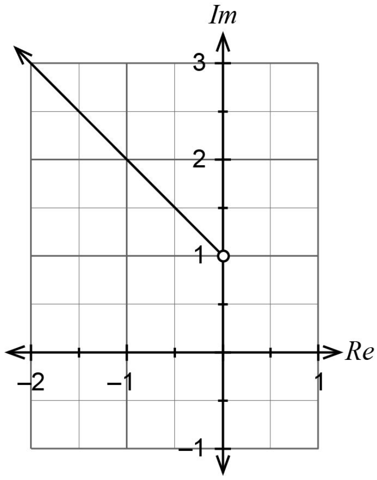
A sketch of the locus of a complex number z is shown below. Write equations or inequalities in terms of z (without using $x = \mathrm{Re}(z)$ or $y = \mathrm{Im}(z)$) for each of the following:

(a) (3 marks)



Solution
$\lvert z-(1-2i)\rvert \le 1$ i.e. $\lvert z-1+2i\rvert \le 1$
Specific behaviours
✓ forms an inequality using the modulus ✓ uses a difference between z and $1-2i$ ✓ uses the radius as 1 unit (the constant on the right-hand side)

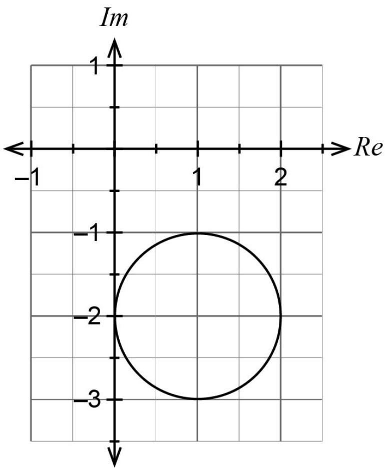
(b) (2 marks)



Solution
$Arg(z-i) = \dfrac{3\pi}{4}$
Specific behaviours
✓ forms an equation stating the argument equals $\dfrac{3\pi}{4}$ ✓ states the argument of $z-i$

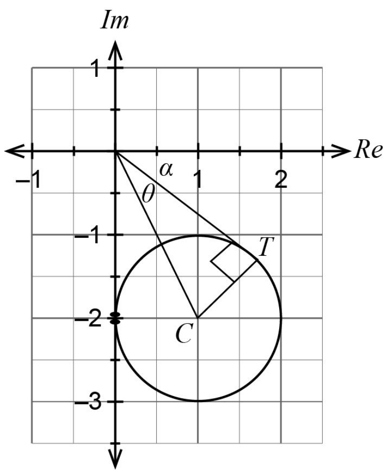
Question 11 (continued)

The sketch in Question 11 part (a) is repeated below, with only the circle indicated.



(c) For the locus from part (a), determine the maximum value for $arg(z)$ correct to 0.01, where $0 \le arg(z) < 2\pi$. (3 marks)

Solution



The tangent to the circle at point T gives the most positive value for $arg(z)$.

i.e. Maximum $arg(z) = 2\pi - \alpha$

In ΔOTC $\sin\theta = \frac{1}{\sqrt{5}}$ $\therefore$ $\theta = 0.4636$

$\tan(\theta + \alpha) = 2$ $\therefore$ $\theta + \alpha = 1.01071$

Hence $\alpha = 1.0107 - 0.4636 = 0.6435$

Hence the maximum value for $Arg(z) = 2\pi - 0.64 = 5.64$ (nearest 0.01).

Specific behaviours

✓ indicates how the maximum value occurs using the tangent
✓ uses appropriate trigonometry to determine θ correctly
✓ evaluates the maximum value for $arg(z)$ correctly

Question 12 **(11 marks)**

The lifetime of an electronic device is distributed as an exponential random variable with mean $\mu = 20$ years and standard deviation $\sigma = 20$ years. A random sample of 50 of these devices is selected. Tam, a graduate electronics engineer, is interested in the mean lifetime $\overline{X}$ of these 50 devices.

(a) State the distribution of the sample mean lifetime $\overline{X}$. Justify your answer. (3 marks)

Solution
$\overline{X}$ is approximately normally distributed as sample size $n = 50 > 30$. $\overline{X} \sim N\left(20, \dfrac{20^2}{50}\right) = N(20,\ 8)$ *i.e.* $\sigma\left(\overline{X}\right) = \sqrt{8} = 2.828$
Specific behaviours
✓ states the sample mean is normally distributed ✓ states the correct mean ✓ states the correct standard deviation (or variance)

(b) Determine the probability that the sample mean lifetime is between 15 and 25 years. (2 marks)

Solution
$P\left(15 < \overline{X} < 25\right) = P\left(\dfrac{15-20}{\sqrt{8}} < z < \dfrac{25-20}{\sqrt{8}}\right)$ $= P(-1.7678 < z < 1.7678)$ $= 0.9229$
Specific behaviours
✓ calculates/uses the correct z scores ✓ determines the correct probability

Jai, the chief engineer, informs Tam that the lifetimes may not be exponentially distributed but could be some more complicated distribution, yet still having mean $\mu = 20$ years and standard deviation $\sigma = 20$ years.

(c) If Jai is correct, will your answer to part (b) change? Explain. (2 marks)

Solution
There will be no change to the answers for the part (b). Since sample size $n = 50 > 30$ then the distribution of sample means is still approximately normal.
Specific behaviours
✓ states that the answers do not change ✓ states the sample mean is still normally distrubuted

Question 12 (continued)

A different random sample of size n of these devices was selected. Repeated sampling with this sample size shows that there is a 3% chance of obtaining a sample mean greater than 25 years.

(d) Determine the value of n. (4 marks)

Solution
Given $P\left(\overline{X} > 25\right) = 0.03$ where $\overline{X} \sim N\left(20, \frac{20^2}{n}\right) = N\left(20, \frac{400}{n}\right)$ $i.e.\ \sigma\left(\overline{X}\right) = \frac{20}{\sqrt{n}}$ If $P(z > k) = 0.03 \quad \therefore\ k = 1.88079...$ $\therefore\ 1.88079... = \frac{25-20}{\frac{20}{\sqrt{n}}}$ i.e. $\frac{20}{\sqrt{n}} = 2.6584...$ Solving gives $n = 56.598...$ i.e. the different sample size was 57 (still $n > 30$)
Specific behaviours
✓ writes a correct probability statement ✓ determines the critical z score to reflect the 0.03 probability ✓ forms an equation relating the z score to the sample size n ✓ solves for the value of n stating an appropriate integer value

Question 13 **(5 marks)**

A particle travels in a straight line so that its velocity v cm/sec and displacement x cm are related by the equation:

$$v = \frac{2}{x}$$

(a) Determine the acceleration a in terms of its displacement x. (2 marks)

Solution
Acceleration $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{4}{x^2}\right)\right) = \frac{d}{dx}\left(\frac{2}{x^2}\right) = 2\left(-2x^{-3}\right)$ i.e. $a = -\frac{4}{x^3}$
Specific behaviours
✓ determines the correct expression for $\frac{1}{2}v^2$ ✓ uses the result $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ correctly to obtain a in terms of x

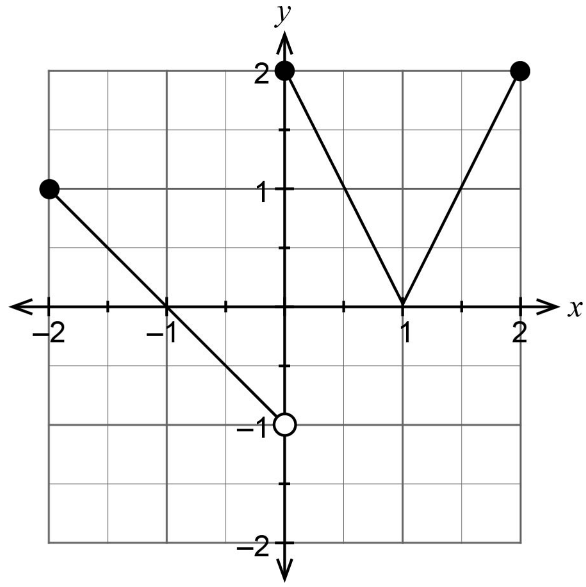
It is known that the initial displacement of the particle is $x = 2$ cm.

(b) Determine the displacement x as a function of time t. (3 marks)

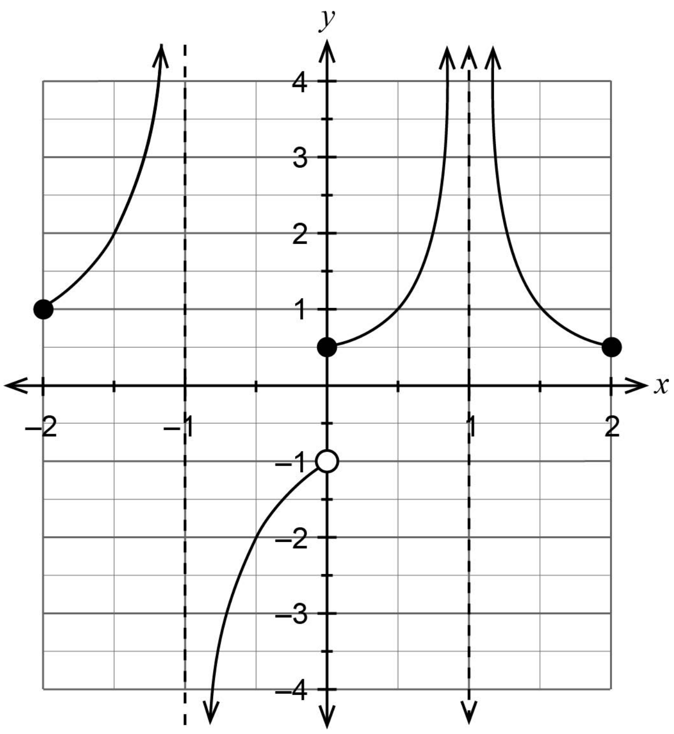
Solution
We have $v = \frac{dx}{dt} = \frac{2}{x}$ $\therefore \int x\,dx = \int 2\,dt$ using separation of variables i.e. $\frac{x^2}{2} = 2t + c_1$ i.e. $x^2 = 4t + c_2$ Using $x(0) = 2$, then $4 = 4(0) + c_2$ Hence $c_2 = 4$ i.e. $x^2 = 4t + 4$ $\therefore x(t) = \sqrt{4t+4}$
Specific behaviours
✓ uses the idea $\frac{dx}{dt} = \frac{2}{x}$ and the separation of variables idea correctly ✓ anti-differentiates correctly ✓ determines the function $x(t)$ correctly

Question 14 **(9 marks)**

The graph of $y = f(x)$ is shown below:

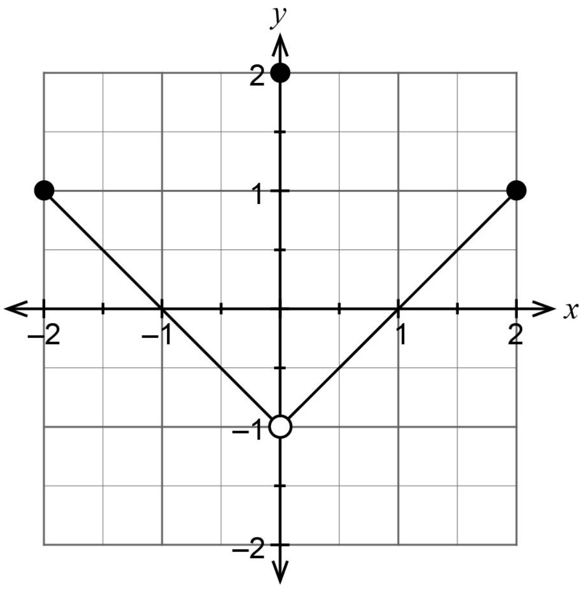


(a) On the axes below, sketch the graph of $y = \dfrac{1}{f(x)}$. (4 marks)



Solution
Shown above.
Specific behaviours
✓ indicates asymptotes at $x=-1,\ x=1$ correctly ✓ indicates ordered pairs $(-2,1)$, $(0,0.5)$, $(2,0.5)$, with a 'hole' at $(0,-1)$ ✓ indicates correct behaviour as $x \to -1$ ✓ indicates correct behaviour as $x \to 1$

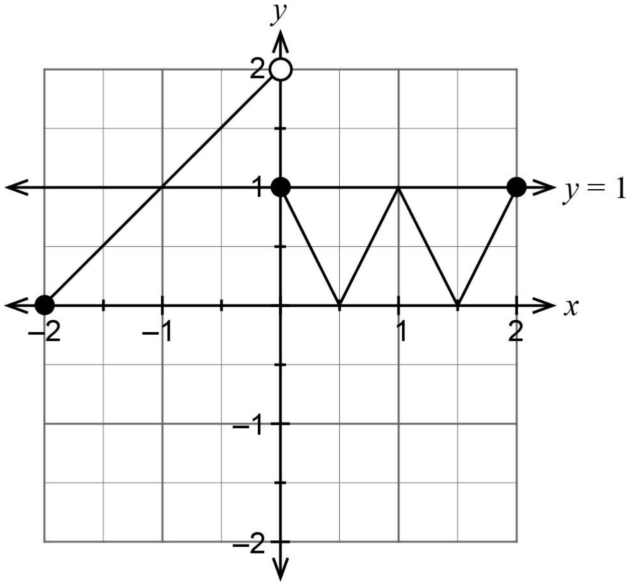
(b) On the axes below, sketch the graph of $y = f(-|x|)$. (3 marks)



Solution
Shown above.
Specific behaviours
✓ indicates the graph of $y = -x - 1$ for $-2 \le x < 0$ ✓ indicates perfect graph symmetry about $x = 0$ ✓ indicates the ordered pair $(0, 2)$

Question 14 (continued)

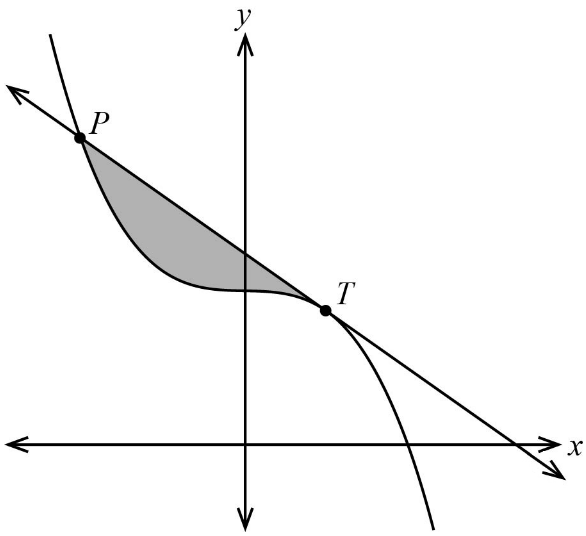
(c) Solve the equation $\left|f\left(x\right)-1\right| = 1$. (2 marks)



Solution
Requires the x intercepts of the graph of $y=\left\|f\left(x\right)-1\right\|$ with $y=1$. From the graph above, $x=-1,\ 0,\ 1,\ 2$
Specific behaviours
✓ considers the intersection of $y=\left\|f\left(x\right)-1\right\|$ with $y=1$ ✓ states the solutions $x=-1,\ 0,\ 1,\ 2$

Question 15 **(8 marks)**

Part of the graph of $x^3 + 8y = 64$ is shown below. A tangent is drawn to the curve at point $T\,(2, 7)$, intersecting the curve again at point P.



(a) Determine the equation of the tangent to the curve at point T. (2 marks)

Solution
Using $x^3 + 8y = 64 \;\therefore\; 3x^2 + 8\dfrac{dy}{dx} = 0 \quad i.e.\; \dfrac{dy}{dx} = -\dfrac{3x^2}{8}$ At $(2,7)$ $m = -\dfrac{3(2)^2}{8} = -\dfrac{3}{2}$ Equation tangent : $y - 7 = -\dfrac{3}{2}(x-2)$ i.e. $y = -\dfrac{3x}{2} + 10$
Specific behaviours
✓ differentiates and evaluates the slope correctly   ✓ forms the equation for the tangent correctly

(b) Determine the area of the shaded region. (3 marks)

Solution
Solving determines point $P\;(-4,16)$ Area of trapped region $= \displaystyle\int_{-4}^{2}\left[\left(-\frac{3x}{2}+10\right)-\left(8-\frac{x^3}{8}\right)\right]dx = \frac{27}{2}$ sq. units
Specific behaviours
✓ forms a definite integral using the correct limits   ✓ writes the correct expression for the integrand as a difference of functions   ✓ evaluates correctly

Question 15 (continued)

The shaded region is then rotated about the x axis.

(c) Calculate the volume of the resulting solid, correct to 0.01 cubic units. (3 marks)

Solution
Volume $V = \int_{-4}^{2} \pi\left(y^2{}_{Line} - y^2{}_{curve}\right)dx = \int_{-4}^{2} \pi\left(10-\frac{3x}{2}\right)^2 - \left(8-\frac{x^3}{8}\right)^2 dx$ $= \frac{2052\pi}{7}$ $= 920.9354 \ldots$ i.e. volume is 920.94 cubic units (nearest 0.01)
Specific behaviours
✓ forms a definite integral using the square of the y values   ✓ forms a definite integral using the factor of π with the correct difference   ✓ evaluates the volume correctly

Question 16 **(9 marks)**

Tom wants to estimate the population mean number of hours, μ, worked by Australians per week. He takes a random sample of 400 workers and determines a 99% confidence interval for μ. The upper limit of this interval is 40.62 hours and the width of this interval is 1.08 hours.

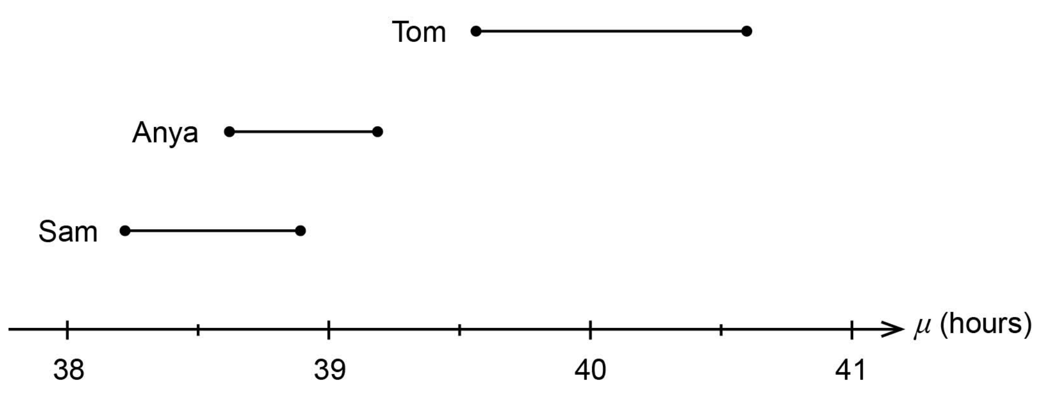
(a) Determine the sample mean for this sample of 400 workers. (2 marks)

Solution
Sample mean $\overline{X} = 40.62 - \dfrac{1.08}{2} = 40.08$
Specific behaviours
✓ uses the correct half-width of the interval ✓ calculates the mean correctly

(b) Calculate, correct to 0.01 hours, the sample standard deviation for the sample of 400 workers. (3 marks)

Solution
Let the sample standard deviation be s. Half-width $0.54 = 2.5758 \times \dfrac{s}{\sqrt{400}}$ Solving gives $s = 4.19$ (2 d.p.)
Specific behaviours
✓ uses the correct z score for 99% confidence ✓ forms the correct equation to relate the interval to the standard deviation ✓ determines the correct standard deviation

Two of Tom's colleagues, Anya and Sam, each take different samples of size 400 and similarly determine 99% confidence intervals for the population mean μ. These confidence intervals, together with Tom's, are shown below.



Question 16 (continued)

(c) Anya makes the following statements based on these confidence intervals. Indicate **why** each of her statements is true or false.

(i) 'Tom's sample has a larger standard deviation compared with that of Sam's and mine.' (1 mark)

Solution
This is true because Tom's confidence interval is wider than the other two.
Specific behaviours
✓ Gives the correct reason for the statement being true.

(ii) 'Tom's method of sampling must be biased since his confidence interval does not overlap with mine or Sam's.' (1 mark)

Solution
This is false because it is possible, due to random sampling, that confidence intervals do not overlap. Not all confidence intervals will contain the true population mean μ.
Specific behaviours
✓ Gives the correct reason for the statement being false.

(d) Which of these three confidence intervals contains the value for μ? Justify your answer. (2 marks)

Solution
We cannot be certain which confidence interval contains the value for μ. This is due to the inherent nature of random sampling.
Specific behaviours
✓ states this cannot be determined ✓ refers to the inherent nature of random sampling

Question 17 **(7 marks)**

Plane Π is represented by the equation: $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

(a) Determine $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and describe what this represents. (1 mark)

Solution
$\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -6 \end{pmatrix}$ (from CAS calculator) This vector is perpendicular to the two vectors in plane Π. Hence this cross product vector is parallel to the NORMAL vector for plane Π.
Specific behaviours
✓ determines the cross product correctly and describes the vector as being the normal for plane Π

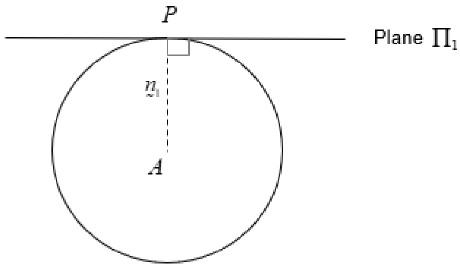
(b) Show that the equation of plane Π can be written as $x - 2y + 2z = 11$. (2 marks)

Solution
Plane Π can also be written in the form : $\underset{\sim}{r} \cdot \underset{\sim}{n}_1 = \underset{\sim}{n}_1 \cdot \underset{\sim}{c}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ i.e. $-3x + 6y - 6z = -9 + 6 - 30$ i.e. $-3x + 6y - 6z = -33$ …. statement (1) $\therefore\ x - 2y + 2z = 11$ (dividing each side by -3)
Specific behaviours
✓ uses the vector form $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \underset{\sim}{c}$ where vector $\underset{\sim}{n}_1$ and $\underset{\sim}{c}$ are used correctly ✓ determines dot products correctly to obtain statement (1) or its equivalent

Question 17 (continued)

Consider sphere S with its centre at point $A\ (3, 4, -1)$.

(c) Determine the Cartesian equation for S if plane Π is tangential to this sphere. (4 marks)

Solution
Consider a diagram showing the perpendicular planes, with the sphere touching plane π at point P.  Equation for the line containing the radius $\overline{AP}$ has direction vector $\underset{\sim}{n}_1$: $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ 2\lambda-1 \end{pmatrix}$ Point P is the intersection of the radial line and the plane Π. Substiting $x = 3+\lambda,\ y = 4-2\lambda,\ z = 2\lambda-1$ into $x-2y+2z = 11$: $(3+\lambda) - 2(4-2\lambda) + 2(2\lambda-1) = 11$ i.e. $9\lambda - 7 = 11$ i.e. $\lambda = 2 \quad \therefore \quad P$ is $(5,0,3)$ Radius $AP^2 = (5-3)^2 + (0-4)^2 + (3+1)^2 = 36$ i.e. radius $AP = \sqrt{36} = 6$ Hence the cartesian equation for sphere S is given by: $(x-3)^2 + (y-4)^2 + (z+1)^2 = 36$.
Specific behaviours
✓ determines the equation for the radial line with direction $\underset{\sim}{n}_1$ ✓ solves to determine the point of intersection P ✓ determines the radius of sphere (or radius squared) ✓ forms the cartesian equation for the sphere correctly

Question 18 **(13 marks)**

A rumour that the Federal Government plans to cut university funding begins to spread around a campus. There is a combined total of 1600 students and staff at this university.

One hundred people know of this rumour via a social media post at 8 am one morning.

Let $N(t)$ = the number of people at the university who have heard the rumour at t hours after 8 am. It is found that the rate at which the rumour spreads is given by the equation:

$$\frac{dN}{dt} = k\,N(1600 - N).$$

At 8 am the rumour was spreading at a rate of 60 people per hour.

(a) Show that $k = 0.0004$. (2 marks)

Solution
Given that $\frac{dN}{dt} = 60$ when $N = 100$
i.e. $60 = k(100)(1500)$ $\therefore$ $k = \frac{60}{(100)(1500)} = \frac{1}{2500} = 0.0004$
Specific behaviours
✓ uses the value 60 people per hour as the value for the derivative ✓ forms the equation to solve for k correctly

(b) At 11 am there were approximately 500 people who had heard the rumour. Using the increments formula, determine the approximate number of people that learn of this rumour between 11 am and 11.15 am. (3 marks)

Solution
At 11.00 am, $t = 3$ and $N = 500$ with $\Delta t = 0.25$ $\Delta N \approx \left(\frac{dN}{dt}\right)\Delta t$ i.e. $\Delta N \approx \left(k(500)(1600-500)\right)\times(0.25)$ $= (220)\times(0.25) = 55$ Hence between 11.00 am and 11.15 am, 55 people learn of the rumour.
Specific behaviours
✓ uses the value $N = 500$ to evaluate the derivative $\frac{dN}{dt}$ ✓ forms the correct expression for ΔN using the correct value for Δt ✓ evaluates ΔN correctly

Question 18 (continued)

(c) Given that $\dfrac{1}{N(1600-N)} = \dfrac{1}{1600}\left(\dfrac{1}{N} + \dfrac{1}{1600-N}\right)$, use the separation of variables technique to show that $N(t)$ is given by $\dfrac{N}{1600-N} = \dfrac{e^{0.64t}}{15}$. (4 marks)

Solution
From $\dfrac{dN}{dt} = 0.0004N(1600-N)$ then $\displaystyle\int \frac{1}{N(1600-N)}\,dN = \int 0.0004\,dt$ $\displaystyle\int \frac{1}{1600}\left(\frac{1}{N} + \frac{1}{1600-N}\right)dN = \int 0.0004\,dt$ …… (1)   i.e. $\dfrac{1}{1600}\left(\ln N - \ln(1600-N)\right) = 0.0004t + c$ …… (2)   i.e. $\ln\left(\dfrac{N}{1600-N}\right) = 1600(0.0004t+c) = 0.64t + k$ i.e. $\dfrac{N}{1600-N} = e^{0.64t}.\,e^{k}$ Using $t=0,\ N=100$ $\dfrac{100}{1600-100} = e^{0}.\,e^{k}$ $\therefore\ e^{k} = \dfrac{100}{1500} = \dfrac{1}{15}$ $\therefore$ $\dfrac{N}{1600-N} = \dfrac{e^{0.64t}}{15}$ as required.
Specific behaviours
✓ separates variables and uses partial fractions to form statement (1) or its equivalent   ✓ integrates correctly to form statement (2) or its equivalent   ✓ obtains expression for $\dfrac{N}{1600-N}$ in terms of an exponential function   ✓ uses the condition $t=0,\ N=100$ to determine the correct constant e^{k}

(d) At what time, correct to the nearest minute, does the rumour spread at the fastest rate? (4 marks)

Solution
Rumour spreads at the greatest rate when $\frac{dN}{dt}$ is at a maximum. This occurs when $N = \frac{1}{2}(1600) = 800$ since $\frac{dN}{dt}$ is a quadratic with roots $N = 0$ and $N = 1600$. Solving $\therefore \quad \frac{800}{1600-800} = \frac{e^{0.64t}}{15}$ yields $1 = \frac{e^{0.64t}}{15}$ From CAS: $t = 4.2313...$ hrs Hence this occurs at 4 hours and 14 minutes after 8.00 am. i.e. At 12.14 pm (to nearest minute) of the same day Note: At this time the rumour spreads at a maximum of 256 people per hour.
Specific behaviours
✓ states that $N = 800$ determines the greatest rate of growth ✓ forms the equation to solve for t ✓ solves correctly for t ✓ states the time of day correctly to the nearest minute

Question 19 **(9 marks)**

A small rocket is fired from the ground at an angle of θ^o to the horizontal with a speed of 70 metres per second. The rocket has the assistance of a steady wind that is blowing horizontally at w metres per second.

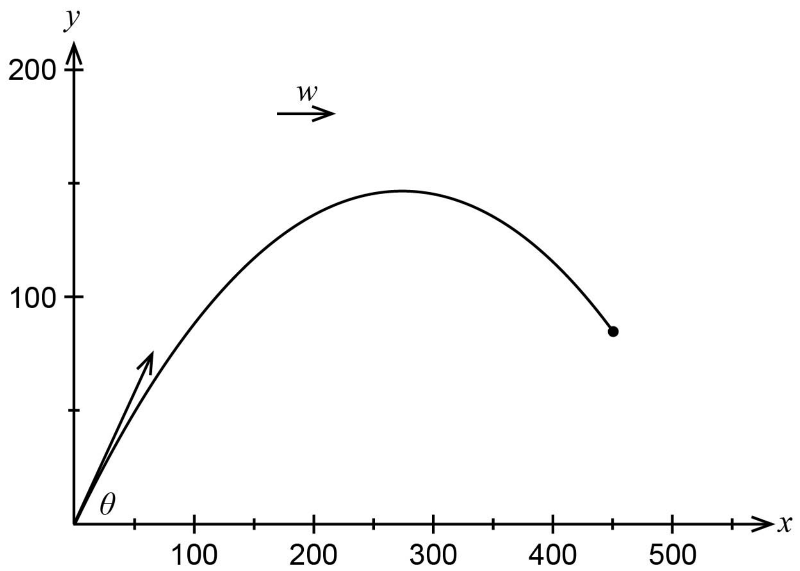
A coordinate system is set up to track the path of the rocket as shown below.

Let t = the number of seconds elapsed after the rocket is fired

$\underset{\sim}{r}(t)$ = the position vector (metres)

$\underset{\sim}{v}(t)$ = the velocity vector (ms^{-1})

$\underset{\sim}{a}(t)$ = the acceleration vector (due to gravity) = $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ (ms^{-2})



(a) Given $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$, show that $\underset{\sim}{r}(t) = \begin{pmatrix} (70\cos\theta + w)t \\ (70\sin\theta)t - 4.9t^2 \end{pmatrix}$. (3 marks)

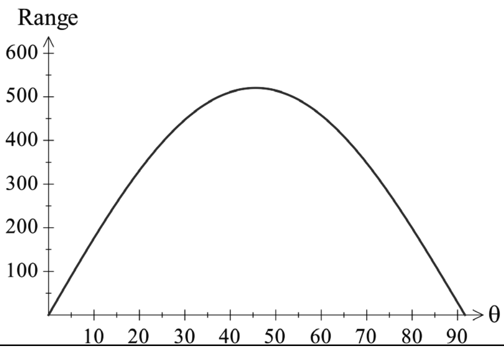
Solution
From $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ then $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt = \begin{pmatrix} k \\ -9.8t + c \end{pmatrix}$
Using $\underset{\sim}{v}(0) = \begin{pmatrix} 70\cos\theta + w \\ 70\sin\theta \end{pmatrix}$ then $\underset{\sim}{v}(t) = \begin{pmatrix} 70\cos\theta + w \\ -9.8t + 70\sin\theta \end{pmatrix}$
From $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt = \begin{pmatrix} (70\cos\theta + w)t \\ -4.9t^2 + (70\sin\theta)t \end{pmatrix}$ since $\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Specific behaviours
✓ anti-differentiates $\underset{\sim}{a}(t)$ correctly using integration constants
✓ uses the correct components for $\underset{\sim}{v}(0)$ to determine $\underset{\sim}{v}(t)$ correctly
✓ anti-differentiates $\underset{\sim}{v}(t)$ and uses $\underset{\sim}{r}(0)$ to obtain $\underset{\sim}{r}(t)$ correctly

(b) Obtain the Cartesian equation for the path of the rocket, in terms of θ and w. (2 marks)

Solution
$\underset{\sim}{r}(t) = \begin{pmatrix} (70\cos\theta + w)t \\ -4.9t^2 + (70\sin\theta)t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ i.e. $t = \dfrac{x}{70\cos\theta + w}$ $\therefore\ y = (70\sin\theta)\left(\dfrac{x}{70\cos\theta + w}\right) - 4.9\left(\dfrac{x}{70\cos\theta + w}\right)^2$ $= \left(\dfrac{70\sin\theta}{70\cos\theta + w}\right)x - \left(\dfrac{4.9}{(70\cos\theta + w)^2}\right)x^2$
Specific behaviours
✓ expresses t correctly in terms of x, θ and w ✓ eliminates t to obtain the cartesian equation for y in in terms of x, θ and w

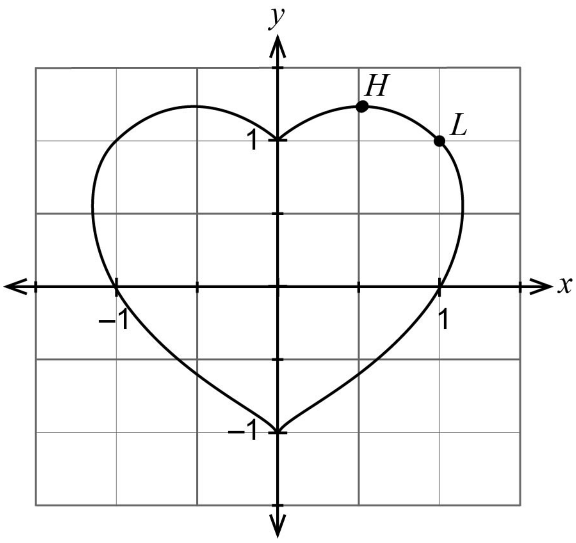
The range of the rocket is defined as the horizontal distance travelled from its launch to the point at which it strikes the ground.

(c) Assuming that the wind speed $w = 2$ metres per second, determine the optimum angle θ so that the range of the rocket is maximised, correct to the nearest 0.1 degree. (4 marks)

Solution
Rocket hits ground when $y = 0$ i.e. $-4.9t^2 + (70\sin\theta)t = 0$ i.e. $t(-4.9t + 70\sin\theta) = 0$ i.e. $t = 0$ or $t = \dfrac{70\sin\theta}{4.9}$ Strikes the ground at $t = \dfrac{70\sin\theta}{4.9}$ i.e. Range $R(\theta) = x = \dfrac{(70\cos\theta + 2)(70\sin\theta)}{4.9}$ Using CAS and plotting the graph for $R(\theta)$ we find that the maximum is at $(45.57^o,\ 520.3)$.      Hence the range will be maximised when $\theta = 45.6^o$ (nearest 0.1 degrees).   Note: the rocket is pointed slightly higher than 45 degrees due to wind assistance.
Specific behaviours
✓ forms the quadratic equation to determine when the rocket strikes the ground   ✓ solves correctly for t (in terms of θ) when the rocket strikes the ground   ✓ formulates the correct expression for the range in terms of θ ✓ determines the correct value for θ (to nearest 0.1 degrees)

Question 20 **(8 marks)**

The graph of $\left(x^2+y^2-1\right)^3 = x^2y^3$ is shown below:



(a) By implicitly differentiating the given equation, obtain an equation relating x, y and $\dfrac{dy}{dx}$. (3 marks)

(Note: Do **not** attempt to obtain $\dfrac{dy}{dx}$ as the subject of this equation.)

Solution
$\dfrac{d}{dx}\left(x^2+y^2-1\right)^3 = \dfrac{d}{dx}\left(x^2y^3\right)$ $3\left(x^2+y^2-1\right)^2.\left(2x + 2y.\dfrac{dy}{dx}\right) = x^2.3y^2.\dfrac{dy}{dx} + 2x.y^3$
Specific behaviours
✓ differentiates the cubed term $\left(x^2+y^2-1\right)^3$ correctly ✓ applies the chain rule to differentiate $\left(x^2+y^2-1\right)$ correctly ✓ applies the product rule correctly to differentiate $\left(x^2y^3\right)$ correctly

(b) Determine the exact slope of the tangent to the curve at the point $L\ (1,\ 1)$. (2 marks)

Solution
Substitute $x=1,\ y=1:\ 3\left(1^2+1^2-1\right)^2.\left(2(1)\ +\ 2(1).\dfrac{dy}{dx}\right)\ =\ 1^2.3(1)^2.\dfrac{dy}{dx}\ +\ 2(1).(1)^3$ i.e. $3\left(2+2\left(\dfrac{dy}{dx}\right)\right)\ =\ 3\left(\dfrac{dy}{dx}\right)+2\qquad \therefore\ 6\ +\ 6\left(\dfrac{dy}{dx}\right)\ =\ 3\left(\dfrac{dy}{dx}\right)\ +\ 2$ …… (1) $\therefore\quad \dfrac{dy}{dx}\ =\ m\ =\ -\dfrac{4}{3}$
Specific behaviours
✓ substitutes $x=1,\ y=1$ into the differential equation to obtain statement (1)   ✓ determines the value for $\dfrac{dy}{dx}$ correctly

At point H on the graph the curve is horizontal.

(c) Determine the coordinates of point H, correct to 0.001. (3 marks)

Solution
Substitute $\dfrac{dy}{dx}=0\ \ 3\left(x^2+y^2-1\right)^2.\left(2x\ +\ 2y.(0)\right)\ =\ x^2.3y^2.(0)\ +\ 2x.y^3$ i.e. $3\left(x^2+y^2-1\right)^2.(2x)\ =\ 2x.y^3$ i.e. $3\left(x^2+y^2-1\right)^2\ =\ y^3$ for $\dfrac{dy}{dx}=0$ Solving simultaneously : $3\left(x^2+y^2-1\right)^2\ =\ y^3$ …… (1) $\left(x^2+y^2-1\right)^3\ =\ x^2y^3$ …… (2)   Using CAS obtains $H\ \ (0.514,\ 1.237)$
Specific behaviours
✓ substitutes $\dfrac{dy}{dx}=0$ correctly into the differential equation   ✓ obtains the equation equivalent to (1)   ✓ uses the original equation to solve simultaneously to 0.001

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